

WORD MAPS IN KAC-MOODY SETTING

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ABSTRACT. The paper is a short survey of recent developments in the area of word maps evaluated on groups and algebras. It is aimed to pose questions relevant to Kac–Moody theory.

Yukimi

Iza yukan

Yukimi ni korobu

Tokoro made.

Snow-viewing

Well, let's go snow viewing,

Till we tumble over.

Matsuo Bashō, 1687

Keywords: word map, simple group, Kac–Moody algebra, Kac–Moody group, algebraic group, Engel words.

1. WORD MAPS

These notes are devoted to problems arising from the general philosophy of word maps. They consist of two sections. In the first one, we describe some recent results and problems related to word maps on simple algebraic groups and finite-dimensional Lie algebras. The objective of the second section is to bring attention to Kac–Moody groups and algebras, look at them from the viewpoint of word maps, and formulate some problems.

The general setting is as follows. Let Θ be a variety of algebras, H be an algebra in Θ , $W(X)$ be a free finitely generated algebra in Θ with generators x_1, \dots, x_n . Fix $w = w(x_1, x_2, \dots, x_n)$ and consider the word map

$$w: H^n \rightarrow H. \quad (1)$$

The map w is the evaluation map: one substitutes n -tuples of elements of the algebra H instead of the variables and computes the value by performing all algebra operations. Varying Θ we arrive at problems on word maps specialized in a fixed Θ . Here we restrict ourselves to considering the varieties of groups and Lie algebras.

The surjectivity of the word map for a given word $w \in W(X)$ and an algebra $H \in \Theta$ is traditionally the central question of the theory. In other words, the question is whether the equation

$$w(x_1, \dots, x_n) = h \quad (2)$$

has a solution for every $h \in H$. However, for many pairs of w and H the phenomenon of surjectivity is just a dream. So the question about the width of H with respect to a given w (see the definition below) often sounds more relevant.

Denote by $w(H)$ the value set of the map w in the algebra H . That is, an element $h \in H$ belongs to $w(H)$ if there exist elements h_1, h_2, \dots, h_n of H such that $w(h_1, h_2, \dots, h_n) = h$. We will be interested in estimating the size of $w(H)$. A reasonable but simpler problem is to compute the span $\langle w(H) \rangle$ of $w(H)$, where span means the verbal subgroup generated by $w(H)$ in the group case and the linear span of $w(H)$ in the case of Lie algebras.

Now assume that Θ is the variety of all groups. As mentioned above, the surjectivity property of a word map is too strong for most pairs (w, H) . Denote by $w(H)^k$ the set of elements $g \in H$ of the form $g = h_1 h_2 \cdots h_k$ where $h_i \in w(H)$, $i = 1, \dots, k$.

The smallest k such that $w(H)^k = H$ is called the w -width of the group H . Denote it by $wd_w(H)$. If H does not have finite w -width, it is said to be of infinite w -width.

We want to estimate how much freedom we have in the general setting of word maps for groups, and to choose a reasonable group H where word maps are evaluated. Let us look at two extreme classes of the variety of all groups: free groups and simple groups. The essence of word maps evaluated at these two poles is quite different.

1.1. Free groups. Let $F_n(X)$, $X = \{x_1, \dots, x_n\}$, be the non-abelian free group with n generators. If we take w to be an arbitrary word in $F_n(X)$, then common sense suggests that $wd_w(F_n(X))$ should be infinite. An important theorem of A. Rhemtulla [99] (see also [104]) confirms that this is indeed the case for all non-universal words, see [104] for details. Moreover, recently A. Myasnikov and A. Nikolaev proved the following.

Theorem 1. [87] *Let H be a non-elementary hyperbolic group. Then the width $wd_w(H)$ is infinite for each non-universal word w .*

Since a “random” group is hyperbolic [89], for “generic” infinite groups the width of any non-universal word is infinite. This result, though disappointing from the viewpoint of word maps, is compensated by a tremendous theory of solutions of equations in free and hyperbolic (non-elementary) groups. For instance, A. Mal’cev [74] described the set of solutions of the equation $[x, y] = [a, b]$, where a, b are generators of the free group and $[x, y] = xyx^{-1}y^{-1}$. L. Comerford–C. Edmunds [31] and R. Grigochuk–P. Kurchanov [41], [42] described all solutions of quadratic equations in free groups. Finally, G. Makanin and A. Razborov described solutions of an arbitrary system of equations over a free group [73], [93]. This theory leads to a developed geometry over free groups and, in its turn, to solution of the famous Tarski problems on elementary theory of free groups, see [58]–[63], [105]–[113], [48], [35], etc.

The theory mentioned above goes far beyond the scope of these notes. We will now turn to another pole of the variety of groups and consider simple groups.

1.2. Algebraic groups. First of all, from the set of all words we choose the following representatives:

Power map:

$$w(x) = x^k. \quad (3)$$

Commutator map:

$$w(x, y) = [x, y] = xyx^{-1}y^{-1}. \quad (4)$$

Engel maps: $w(x, y) = e_n(x, y)$.

$$e_1(x, y) = [x, y], \dots, e_{n+1}(x, y) = [e_n, y] = [x, y, \dots, y]. \quad (5)$$

Quasi-Engel maps: $w(x, y) = v_n(x, y)$.

$$v_1(x, y) = x^{-2}y^{-1}x, \dots, v_{n+1}(x, y) = [xv_n(x, y)x^{-1}, yv_n(x, y)y^{-1}]. \quad (6)$$

Quasi-Engel maps: $w(x, y) = s_n(x, y)$.

$$s_1(x, y) = x, \dots, s_{n+1}(x, y) = [y^{-1}s_n(x, y)y, s_n(x, y)^{-1}]. \quad (7)$$

Engel maps are related to nil elements and nilpotency, quasi-Engel maps do the same with respect to solvability [127], [10], [18], [43].

We start with a particular problem, which seems to be, at the moment, the most challenging and tempting among the problems on word maps for semisimple algebraic groups.

Conjecture 2. *Let $G = PSL_2(\mathbb{C})$, and let $w = w(x, y)$ be an arbitrary non-identity word in $F(x, y)$. Then the word map $w: PSL_2(\mathbb{C}) \times PSL_2(\mathbb{C}) \rightarrow PSL_2(\mathbb{C})$ is surjective. In other words, the equation*

$$w(x_1, x_2) = a$$

has a solution for every $a \in PSL_2(\mathbb{C})$.

Although this conjecture is widely open, there are several partial results. First of all, note that the power map $w = x^n$ is surjective on $PSL_2(\mathbb{C})$, since all roots are extractable in this group. The next class of surjective words is given by

Proposition 3. *Suppose that a word $w(x, y) = x^{i_1}y^{j_1}x^{i_2}y^{j_2}\dots x^{i_k}y^{j_k}$ is not an identity of the infinite dihedral group D_∞ . Then the word map $w(x, y)$ is surjective on $PSL_2(\mathbb{C})$.*

Proof. We can assume that $(i_1 + \dots + i_k) + (j_1 + \dots + j_k) = 0$. Otherwise, setting $x = y$, we arrive at the power word x^m . The equation $x^m = c$ is solvable, since in $PSL_2(\mathbb{C})$ one can extract roots of an arbitrary degree. So, assuming the condition above, we need to solve the equation $w(x, y) = c$ in $PSL_2(\mathbb{C})$. Let us plug two involutions a, b of $PSL_2(\mathbb{C})$ into the word w . Since $a^{-1} = a$, $b^{-1} = b$, we have $w(a, b) = (ab)^\ell$, $1 \leq \ell \leq k$, or $w(a, b) = (ba)^\ell$, $1 \leq \ell \leq k - 1$. Note that $ab \neq 1$ because w is not an identity in D_∞ . Extracting roots, we arrive at a system of equations:

$$a^2 = 1, \quad b^2 = 1, \quad ab = c', \quad (8)$$

where c' is a prescribed element of $PSL_2(\mathbb{C})$ (explicitly, c' is either an ℓ th root of c or an ℓ th root of bc). It remains to notice that in $PSL_2(\mathbb{C})$ every element is a product of two involutions [13]. \square

Corollary 4. *All n -Engel maps are surjective on $PSL_2(\mathbb{C})$.*

Proof. Let $a, b \in PSL_2(\mathbb{C})$ be involutions. We have $e_1(a, b) = aba^{-1}b^{-1} = abab = (ab)^2$. Show that $e_n(a, b) = (ab)^{2^n}$. By induction,

$$\begin{aligned} e_n(a, b) &= [e_{n-1}(a, b), b] = [(ab)^{2^{n-1}}, b] = (ab)^{2^{n-1}} b (ba)^{2^{n-1}} b = \\ &= (ab)^{2^{n-1}} (ab)^{2^{n-1}} = (ab)^{2^n}. \end{aligned}$$

□

Among other interesting word maps which fall under the conditions of Proposition 3 is the family $w_1(x, y) = [x, y]$, $w_{n+1}(x, y) = [w_n(x, y), \delta_n(x, y)]$, where $\delta_n(x, y) = y$ if $n = 2k$ or $\delta_n(x, y) = x$ if $n = 2k + 1$. Moreover, T. Bandman proved

Theorem 5. [5] *The word map defined by $w(x, y) \in F^{(1)} \setminus F^{(2)}$ is surjective on $PSL_2(\mathbb{C})$.*

Here $F^{(1)} = [F, F]$ and $F^{(2)} = [F^{(1)}, F^{(1)}]$ are the first and the second terms of the lower central series of the two-generated free group $F = F(x, y)$. The proof is reduced to verification of the surjectivity of $w(x, y)$ on the unipotent elements. Note that the surjectivity of $w(x, y)$ on the semisimple elements of $PSL_2(\mathbb{C})$ can be derived by computations of trace polynomials (see, for example, [40]) and application of methods of [11], [8]. Another proof of the surjectivity of $w(x, y)$ on the semisimple elements can be found in [56].

From Theorem 5 it follows that a key to solution of Conjecture 2 lies in behaviour of words w sitting in $F^{(2)}$ or deeper in the lower central series. In this sense a result of [5] stating that the quasi-Engel maps $w(x, y) = s_n(x, y)$ are surjective for any $n \geq 1$ is of promising importance. Yet other computations from the same paper show that the word

$$w(x, y) = [[x, [x, y]], [y, [x, y]]] \in F^{(2)} \quad (9)$$

is surjective on $PSL_2(\mathbb{C})$ as well.

Conjecture 2 is a particular case of [54, Question 2]. We repeat this question here:

Problem 6. Let \mathcal{G} be the class of simple groups G of the form $G = \mathbb{G}(k)$ where $k = \bar{k}$ is an algebraically closed field and \mathbb{G} is a semisimple adjoint linear algebraic group. Is it true that word maps are surjective for all non-power, non-trivial words?

It should be noted that the power map $w = x^n$ is surjective on an arbitrary $G = \mathbb{G}(k)$ if n is prime to 30 (see [121], [27]–[30], [66]). Moreover, the only simple algebraic adjoint group with extracting arbitrary roots is of type A_n . Hence, Problem 6 for the groups of type A_n can be reformulated as follows:

Problem 7. Is the word map $w: PSL_n(\mathbb{C}) \times PSL_n(\mathbb{C}) \rightarrow PSL_n(\mathbb{C})$ surjective for any non-trivial $w(x, y) \in F_2(x, y)$?

Despite many efforts applied to Problem 6 within past years there are only few developments for specific word maps. For instance, the commutator map $w(x, y) = [x, y]$ is surjective on every semisimple adjoint linear algebraic group $G = \mathbb{G}(k)$ over an algebraically closed field k [94] (cf. the Ore conjecture for finite simple groups: every element of a group is a commutator). The next theorem concerning Engel words is due to N. Gordeev.

Theorem 8. [39] *The image of the Engel map $e_n(x, y)$ on a semisimple adjoint linear algebraic group $G = G(k)$, $k = \bar{k}$, contains all semisimple and all unipotent elements.*

So, for elements $g = tu$ that have the Jordan form with non-trivial semisimple part t and non-trivial unipotent part u , it is unclear whether they are covered by $e_n(x, y)$. In the same paper it is proved that the Engel map is surjective on $PSp_4(k)$ and $G_2(k)$. Summing up, the surjectivity of Engel words is unknown for all groups except $PSL_2(k)$, $PSp_4(k)$ and $G_2(k)$. For quasi-Engel maps the only treated case is $PSL_2(k)$ [5].

Now we are interested in estimating $wd_w(G)$, where $w(x_1, \dots, x_n)$ is any word in $F(X)$. The amazing theorem of A. Borel [17] (see also [67], [54]) states that the image of w on every appropriate algebraic group is large in Zariski topology.

Theorem 9. [17] *Let $w(x_1, \dots, x_n)$ be a non-trivial word, let G be a connected semisimple linear algebraic k -group, and let k be an arbitrary field. The image of the word map $w: G^n(k) \rightarrow G(k)$ is Zariski dense.*

The next corollary confirms that for an algebraically closed field the image is indeed big.

Corollary 10. *Let k be an algebraically closed field. Then $w(G(k))^2 = G(k)$.*

The latter fact means that the width of any word map evaluated on a connected semisimple algebraic group over an algebraically closed field is less than or equal to two.

1.3. Algebraic groups. Forms. For an extended discussion of the situation with word maps in this case see [66]. Here we just mention a few principal results.

First of all, the power word map x^n is surjective for the compact real forms of all semisimple algebraic groups G (cf. [30]). The fact that the commutator map $w(x, y) = [x, y]$ is surjective for the case of compact Lie groups $G = SU_n(\mathbb{R})$ goes back to H. Tôyama and M. Gotô. The same is true for \mathbb{R} -anisotropic simple algebraic groups $G = G(\mathbb{R})$, see [126], [40]. In contrast with the split case, the behaviour of Engel word maps on anisotropic groups is well-known due to A. Elkasapy–A. Thom (cf. also [39]):

Theorem 11. [33] *Let G be an \mathbb{R} -anisotropic simple algebraic group. Then for every $n \geq 1$ the n -Engel word map $e_n(x, y): G^2 \rightarrow G$ is surjective.*

In particular, e_n is surjective on compact Lie groups $SU_n(\mathbb{R})$. Nothing is known about the behaviour of quasi-Engel maps. The general result similar to that of Theorem 5 was earlier established by A. Elkasapy–A. Thom [33] for compact groups: if $w(x, y)$ does not belong to $F^{(2)}$, then the corresponding word map is surjective on $SU_n(\mathbb{R})$.

Remark 12. The compact counterparts of Conjecture 2 and Problem 6 have negative answers [124]. From the construction of A. Thom it also follows that the width $wd_w(G)$ may be infinite in the compact case. The reason is that for some words w the image of the corresponding word map is a small neighbourhood of the identity matrix, in particular, is not dense in Euclidean topology. The authors are not aware of any example of a non-power word w (i.e., w not representable as a proper power of another word) such that the corresponding map on a simple real algebraic group would be dominant in Euclidean topology but not surjective.

1.4. Finite simple groups. New achievements in the theory of word maps evaluated on finite simple groups may be considered as a sort of catalyst which gave rise to ongoing interest in this theory.

Among jewels of the last years was a positive solution of Ore’s problem: every element of a finite simple group is, indeed, a single commutator (see [90], [34] and the final solution in [71]; a nice survey is given in [75]).

A formidable progress in the description of images of word maps for finite simple groups was obtained by M. Larsen and A. Shalev, who stimulated a development of this area of research under the name of “Waring type problems”. The final result of Larsen–Shalev–Tiep is as follows:

Theorem 13 ([69]). *Let w be an arbitrary non-trivial word of $F(x_1, \dots, x_n)$. There exists a constant $N = N(w)$ such that for all non-abelian simple groups of order greater than N one has*

$$w(G)^2 = G.$$

So, for every w the w -width of a finite simple group is less than or equal to two. We believe that the coincidence of widths in Borel’s theorem for algebraic groups and the theorem of Larsen–Shalev–Tiep quoted above might have a conceptual explanation in the spirit of principles of model theory: loosely speaking, a statement on algebraic varieties formulated over an algebraically closed field should have a counterpart over every sufficiently large finite field. Classical theorems of Ax–Kochen–Grothendieck can serve as instances of such a principle, see [114]. This approach is awaiting an interested reader. We also refer to the papers [117]–[119], [69], [68], [8], [7], [54], [88], [77], [103], [49], [57], [70], etc., for details, surveys and further explanations.

Returning to specific word maps evaluated on finite simple groups, even the case of Engel words $e_n(x, y)$ is still widely open. The only known result is the surjectivity of e_n on $PSL_2(\mathbb{F}_q)$ obtained in the paper of Bandman–Garion–Grunewald.

Theorem 14. [7] *Let $G = PSL_2(\mathbb{F}_q)$. The n -th Engel word map $e_n(x, y): G^2 \rightarrow G$ is surjective provided that $q > q_0(n)$. If $n \leq 4$, then $e_n(x, y)$ is surjective for every q .*

The main ingredient of the proof is dynamics of trace maps, see [8].

1.5. Finite-dimensional simple Lie algebras. In this case $W(X)$, $X = \{x_1, \dots, x_n\}$, is the finite-dimensional free Lie algebra, that is, the algebra of Lie polynomials with coefficients from a field k . We will consider word maps $w: L^n \rightarrow L$ where $w \in W(X)$, L is a split semisimple Lie algebra.

Let us treat the brackets in maps (4) as Lie operations. The role of quasi-Engel maps (6), (7) is played by the sequence

$$v_1(x, y) = [x, y], \dots, v_n(x, y) = [[v_{n-1}(x, y), x], [v_{n-1}(x, y), y]], \quad (10)$$

which is related to the solvability property of Lie algebras [6], [43].

In this setting, many problems on word maps formulated above for groups have solutions for Lie algebras. For example, these are the Ore problem [19] and the problem of the surjectivity of Engel word maps [9]. The analogue of Conjecture 2 has a partial solution [56]. Namely, the image of any non-trivial homogeneous $w \in W(X)$ evaluated on $\mathfrak{sl}_2(k)$, where k is an algebraically closed field, is either 0, or the set of all non-nilpotent traceless matrices, or $\mathfrak{sl}_2(k)$. Note that the trivial

image can appear, since for simple Lie algebras w can be an identity of L . Further, the Lie polynomial

$$v_2(x, y) = [[[x, y], x], [[x, y], y]] \quad (11)$$

covers only the non-nilpotent values on $\mathfrak{sl}_2(k)$ [9], realizing the second possibility for images of an arbitrary word w . In particular, this means that the quasi-Engel word $w(x, y)$ is not surjective on $\mathfrak{sl}_2(k)$. The proof in the Lie case is a variation of a similar general result on multilinear polynomials evaluated on simple associative algebras, which is due to Kanel-Belov–Malev–Rowen:

Theorem 15. [55] *If w is a multilinear polynomial evaluated on the matrix ring $M_2(k)$ (where k is a quadratically closed field), then the image of w is either $\{0\}$, or k , or $\mathfrak{sl}_2(k)$, or the full matrix algebra $M_2(k)$.*

The question whether a multilinear Lie polynomial w can take only the values 0 or $\mathfrak{sl}_2(k)$ on $\mathfrak{sl}_2(k)$ is still open. We will finish this brief overview with presenting a Lie-algebraic counterpart of Borel’s theorem:

Theorem 16. [9] *Let L be a split semisimple Lie algebra. Suppose that a Lie polynomial $w(x_1, \dots, x_n)$ is not an identity of the Lie algebra $\mathfrak{sl}_2(k)$. Then the image of $w: L^n \rightarrow L$ is Zariski dense.*

Remark 17. It is unclear whether it is possible to improve the result by suppressing the condition on $w(x_1, \dots, x_n)$ in Theorem 16. Also, the case of word maps evaluated on non-classical Lie algebras is generally open (see [54] for further discussions). If the base field k is not algebraically closed and L is not split, the situation is not understood either. Even for the simplest word $w = [x, y]$, even for $k = \mathbb{R}$, there are only partial results: the corresponding map $L^2 \rightarrow L$ is surjective if L is a compact simple algebra [46], [2] (D. Akhiezer [2] also treats some non-compact algebras). Surprisingly, for $w = [x, y]$ there is a very nice result of “Waring type” due to G. Bergman–N. Nahlus [15] who use recent two-generation theorems by J.-M. Bois [16]: if L is a finite-dimensional simple Lie algebra defined over any infinite field of characteristic different from 2 and 3, then every $a \in L$ is a sum of two brackets: $a = [x, y] + [z, t]$. (Over \mathbb{R} , a simple proof can be found in [46]; see [38] for the case of arbitrary classical Lie algebras.) To the best of our knowledge, there are no examples of simple Lie algebras of infinite bracket width; moreover, we do not know any example where one bracket is not enough.

In the case where the base field k is replaced with some ring R , the situation is almost totally unexplored, see [65] for discussion of some cases where R is of arithmetic type.

2. WORD MAPS FOR KAC-MOODY GROUPS AND ALGEBRAS. PROBLEMS

Our next aim is to discuss how the results of the previous section can be treated from the viewpoint of Kac–Moody groups and algebras. Let $A = (a_{ij})$, $1 \leq i, j \leq n$, be a generalized Cartan matrix, that is, $a_{ii} = 2$, $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} = 0$ if and only if $a_{ji} = 0$. Every $A = (a_{ij})$ gives rise to a complex Kac–Moody algebra \mathfrak{g}_A , see [79], [50], etc. According to the type of A (positive-definite, positive-semidefinite, indefinite), we arrive at finite-dimensional simple Lie algebras, affine Kac–Moody algebras, and indefinite Kac–Moody algebras, respectively.

2.1. Minimal (incomplete) Kac–Moody groups. Split case. Given a generalized Cartan matrix A and a field k (or a ring R), the value $G_A(k)$ of the Tits functor $G_A: \mathbb{Z}\text{-Alg} \rightarrow \text{Grp}$ defines a minimal Kac–Moody group over k , see [125] (cf. [26], [84]). One can view this functor as a generalization of the Chevalley–Demazure group scheme. We assume that A is indecomposable. As a rule, we assume that the functor G_A is simply connected. (However, speaking about the simplicity of a Kac–Moody group $G_A(k)$ (resp. k -algebra \mathfrak{g}_A), we will freely, often without special mentioning, use common language abuse, assuming that we go over to its subquotient, taking the derived subgroup (resp. subalgebra) and factoring out the centre, if necessary.)

If A is a definite matrix, the group $G_A(k)$ is a Chevalley group $G_\Phi(k)$ where Φ is the root system corresponding to A . Word maps arising in this case were considered in the previous section.

If A is of affine type, $G_A(k)$ is isomorphic to the Chevalley group $G_\Phi(k[t, t^{-1}])$ where $k[t, t^{-1}]$ is the ring of Laurent polynomials. Since $G_A(k)$ is a Chevalley group over a ring, all Borel-type considerations are irrelevant. However, the width of $G_A(k)$ with respect to natural words is unknown.

Problem 18. What is the width of the affine Kac–Moody group $G_A(k) \simeq G_\Phi(k[t, t^{-1}])$ with respect to commutators? Is this width finite?

A comprehensive survey of the widths of the Chevalley groups over rings can be found in [44], [123], [122]. We shall quote some facts from these sources. First of all, since $k[t, t^{-1}]$ is a Euclidean ring, the group $G_\Phi(k[t, t^{-1}])$ is perfect and, moreover, $G_\Phi(k[t, t^{-1}]) = E_\Phi(k[t, t^{-1}])$ where $E_\Phi(k[t, t^{-1}])$ is the elementary subgroup. Further, it is worth taking into account an unexpected result from [44] which states that finite width in commutators is equivalent to bounded generation in elementary unipotent elements (see [86] for a survey of the latter question). Furthermore, there is an example by R. K. Dennis and L. Vaserstein:

Theorem 19 ([32]). *The group $SL(3, \mathbb{C}[t])$ does not have finite width with respect to commutators.*

This example shows that the commutator width for Chevalley groups over polynomial rings with coefficients from \mathbb{C} is infinite. It also prevents from predictions in the case of Laurent polynomials, without clarifying the situation with usual polynomial rings.

Remark 20. In general, the question whether a Chevalley group $G_\Phi(k[t])$ over a polynomial ring with coefficients in k has a finite commutator width should depend on the choice of the ground field k . Most likely, for fields of infinite transcendence degree over the prime subfield the negative answer looks plausible. To the contrary, for finite fields an affirmative answer sounds quite reasonable. We quote [122]: “It is amazing that the answer is unknown already for $SL_n(F[x])$, where F is a finite field or a number field” and thank A. Stepanov for the whole remark.

Let now $G_A(k)$ be a Kac–Moody group of indefinite type. All these groups are perfect for fields of size > 3 . Moreover, B. Rémy [97] and P.-E. Caprace–B. Rémy [22] showed that the minimal indefinite adjoint Kac–Moody groups $G_A(F_q)$ are simple provided $q > n > 2$ for every A and for $n = 2$ for some A . J. Morita and B. Rémy [85] proved that in the case where k is the algebraic closure of \mathbb{F}_q these

groups are simple. However, the simplicity problem for minimal groups over other fields is open.

P.-E. Caprace and K. Fujiwara [20] showed that over finite fields these (infinite) simple groups have infinite commutator width. In general, especially over the \mathbb{C} and \mathbb{R} , problems on word maps are not yet settled.

Problem 21. Study the image of word maps for minimal non-affine Kac–Moody groups over various fields. In particular, what can be said about the commutator width and power width? Are there some density properties? Are there analogues of results of Waring type for arbitrary words?

The first step is to check what tools that proved to be useful in finite-dimensional case can be extended to Kac–Moody setting. This is not completely hopeless, as is shown in the papers [82], [83], where this was done for Gauss decomposition.

2.2. Maximal (complete) Kac–Moody groups. Split case. Let $G_A(k)$ be an incomplete Kac–Moody group. There are several ways to complete this group with respect to an appropriate topology. Different methods of completions lead to very similar groups (Rémy–Ronan groups [98], Mathieu groups [78], Carbone–Garland groups [24]) whose distinctions are not very important for our goals. It is known that the Rémy–Ronan group is smallest among them and can be obtained as a quotient of others. So, once we are talking about a simple (indefinite) Kac–Moody complete group we can assume that this is one of the incarnations of the Rémy–Ronan group. Denote it by $\overline{G}_A(k)$. This group is simple (both as a topological group and an abstract group), see [80], [52], [37], [23], [76], [102], [21], [97], etc.

Let first $\overline{G}_A(k)$ be a complete affine Kac–Moody group. Then $\overline{G}_A(k)$ is isomorphic to a Chevalley group of the form $G_\Phi(k((t)))$ where $k((t))$ is the field of formal Laurent series over k . Thus, $\overline{G}_A(k)$ modulo centre is a simple algebraic group. Here is an immediate corollary of Borel type:

Corollary 22. *Let $\overline{G}_A(k)$ be a complete adjoint affine Kac–Moody group, and let $w = w(x_1, \dots, x_n)$ be an arbitrary non-trivial word. Then the image of $w: \overline{G}_A(k)^n \rightarrow \overline{G}_A(k)$ is Zariski dense.*

Note that the field $K = k((t))$ is not algebraically closed. Thus, it is not hard to prove that, say, if $w = x^n$ is a power word, then the induced word map is not surjective, both for $G = SL_2$ defined over $k((t))$ and $k[t, t^{-1}]$: it is enough to look at the equation $x^n = \text{diag}(t, t^{-1})$.

Further, Corollary 22 does not provide estimates for particular $wd_w(G_\Phi(K))$. However, recently Hui–Larsen–Shalev [47] proved that for an arbitrary w we have $wd_w(G_\Phi(K)) \leq 4$ in every simple Chevalley group over any infinite field K . Thus, $wd_w(\overline{G}_A(k)) \leq 4$. In a similar spirit, from [47] it follows that the width of $\overline{G}_A(k)$ in squares equals 2. So, one can ask the following question:

Question 23. What is the commutator width of $\overline{G}_A(k)$?

Let now $\overline{G}_A(k)$ be an arbitrary complete Kac–Moody group. The problem whether a suitable variant of Borel’s theorem is valid for $\overline{G}_A(k)$ is among the most intriguing ones. For example, let us view $\overline{G}_A(k)$ as a simple algebraic ind-scheme [64], [78], [101] (see also [1], [51], etc.). Then $\overline{G}_A(k)$ is endowed with a Zariski-type topology [115], [116], [53]. One of the possible approaches is to test the image of a word map in this topology.

Problem 24. Let $\overline{G}_A(k)$ be a complete non-affine Kac–Moody group, and let $w = w(x_1, \dots, x_n)$ be an arbitrary non-trivial word. Is it true that $w((\overline{G}_A(k))^n)$ is dense in a Zariski topology?

Here are some related questions.

Questions 25. Is there a bound for $wd_w(\overline{G}_A(k))$? What is the commutator width of $\overline{G}_A(k)$? How big is the image $w((\overline{G}_A(k))^n)$ in the natural topology of $\overline{G}_A(k)$?

In a more general context, one can try to investigate an eventual gap between the density and surjectivity properties:

Question 26. Do there exist a locally compact topological group G , simple at least as a topological group, and a word $w = w(x_1, \dots, x_n)$ non-representable as a proper power of another word such that the corresponding word map $w: G^n \rightarrow G$ is not surjective but the image of w is dense?

Kac–Moody groups seem to be a sufficiently rich testing ground for such sort of questions.

2.3. Forms of Kac–Moody groups. The theory of forms for Kac–Moody groups is mainly developed in [95], [96], [101], [102], [100], [45], [91], [92], etc. All questions stated before in the split case also make sense for twisted forms. In particular, for affine real forms classified in [14], this is related to commutator widths in twisted Chevalley groups over the ring of Laurent polynomials $R[t, t^{-1}]$. Another question to study in the compact case is the phenomenon of Thom’s sequence (see Section 1): can it occur for word maps on any compact Kac–Moody group?

The case of integer forms of Kac–Moody groups, which were extensively studied both from arithmetic viewpoint (Eisenstein series, Langlands program) and with an eye towards applications in mathematical physics (superstrings, supergravity), see, e.g., [36], [24], [12] and the references therein, is even less explored. Any question on the word width of such groups seems challenging. In a more group-theoretic spirit, one can mention the papers of D. Allcock and L. Carbone [3], [4], who established some finite presentability results in the affine and hyperbolic cases.

2.4. Kac–Moody algebras. As in Section 1, one can ask questions on the image of the map induced by a Lie polynomial, as well as questions of width type, for Kac–Moody algebras, *mutatis mutandis*. For example, it seems natural to proceed in the spirit of Remark 17 and look at the bracket width of simple (subquotients of) algebras \mathfrak{g}_A . Can it happen that such an algebra is of infinite bracket width? Of bracket width greater than one?

In the case that for some simple algebra the bracket map turns out to be surjective, one can study n -Engel maps ($n \geq 2$) and maps induced by general multilinear Lie polynomials.

Another problem is to find out whether some counterpart of Borel-type Theorem 16 exists in Kac–Moody setting.

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Three co-authors greet Jun Morita on his 60th birthday and wish him good health and activity.

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